A convolution inequality, yielding a sharper Berry-Esseen theorem for summands Zolotarev-close to normal

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Theorem 1. *Let F_1, F_2, H_1, H_2 be probability distribution functions on $\mathbb{R}$, with H_1, H_2 having finite Lipschitz constants $\|H_1'\|_\infty, \|H_2'\|_\infty$. Then we have*

$$\|F_1 \star F_2 - H_1 \star H_2\|_\infty \quad \le \quad \left(\sqrt{\|H_2'\|_\infty \|F_1 - H_1\|_1} + \sqrt{\|H_1'\|_\infty \|F_2 - H_2\|_1}\,\right)^2 .$$

Corollary. *Let $P \in \mathcal{P}_2$. Then* $\left\|\widetilde{P^{*2}} - \mathrm{N}\right\|_{\mathrm{K}} \;\le\; \frac{4}{\sqrt{2\pi}}\zeta_1(\widetilde{P} - \mathrm{N}).$

Theorem 2. *There exists a constant $c \in\,]0,\infty[$ satisfying*

$$\left\|\widetilde{P^{*n}} - \mathrm{N}\right\|_{\mathrm{K}} \;\le\; \frac{c}{\sqrt{n}}\zeta_1(\widetilde{P} - \mathrm{N}) \vee \zeta_3(\widetilde{P} - \mathrm{N}) \qquad \textit{for } P \in \mathcal{P}_3 \textit{ and } n \ge 2.$$

Notation: $F \star G :=$ distribution function of convolution $P * Q$ of corresponding laws P, Q. $\mathcal{P}_r :=$ set of all non-Dirac laws on $\mathbb{R}$ with finite rth moments, $\widetilde{P} :=$ standardisation of $P \in \mathcal{P}_2$, $\mathrm{N} :=$ standard normal law, Kolmogorov's distance $\|P - Q\|_{\mathrm{K}} := \|F - G\|_\infty$, Zolotarev's (1976) $\zeta_r(\widetilde{P} - \widetilde{Q}) := \sup\{\int f \,\mathrm{d}(\widetilde{P} - \widetilde{\mathrm{Q}}) : f^{(r-1)}$ Lipschitz with constant $1\}$, $\zeta_1(P - Q) = \|F - G\|_1$.

Motivation: Three up to now optimal (incomparable) improvements of the classical Berry-Esseen bound, namely vanishing if $\widetilde{P} = \mathrm{N}$, are due to Sazonov (1972) - Zolotarev (1973), Ulyanov (1976), Senatov (1998). Theorem 2 is simpler and strictly sharper than any of these.

Proof of Thm. 1 uses a non-standard extreme point reduction to prepare for Cauchy-Schwarz. The corollary follows using $\|\widetilde{P^{*2}} - \mathrm{N}\|_{\mathrm{K}} = \|\widetilde{P}^{*2} - \mathrm{N}^{*2}\|_{\mathrm{K}}$, and then simply yields Theorem 2 by another theorem of Zolotarev (1986, 1997), which is hence stronger than it looked so far. □